Anomalous contribution and fluctuation theorems of perturbed diffusion processes

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(Presentation based on joint works with Xiao Jin[1])

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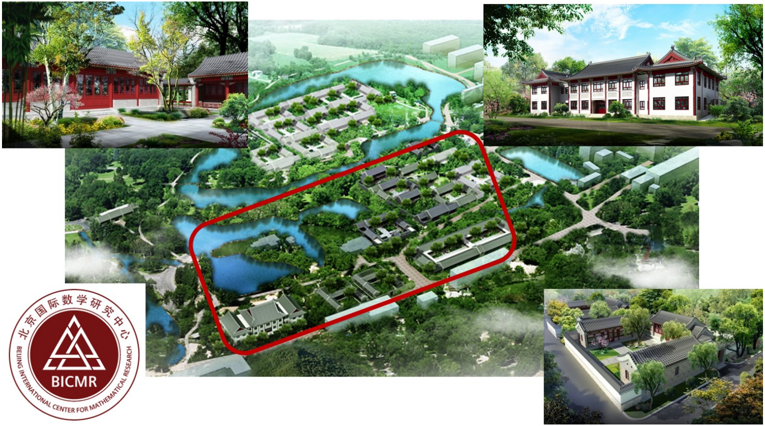
BICMR : Beijing International Center for Mathematical Research



BIOPIC : Biomedical Pioneering Innovation Center

Research in Ge's group

Stochastic mathematical physics, stochastic biophysics and biomathematics/biostatistics

- Mathematical theory driven by applications from physics, chemistry or biology
  - Stochastic theory of nonequilibrium thermodynamics and statistical mechanics (JSP06,08,17 ;PRE09,10,13,14,16,18 ;JCP12 ;JSTAT15) ;
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  - Nonequilibrium landscape theory and rate formulas for single-molecule and single-cell biology (PRL09,15 ;JRSI11 ;Chaos12 ;PlosCB18) ;
- Applications of mathematics and statistics to solve scientific problems in chemistry or biology
  - Stochastic modeling in systems biology and biophysical chemistry (JPCB08,13,16 ;JPA12 ;SPA17 ;PR12 ;Science13 ;Cell14 ;MSB15) ;
  - Statistical machine learning of single-cell data

Brief history of fluctuation theorems

- The discovery of fluctuation theorems is one of the major breakthroughs in the field of nonequilibrium statistical mechanics in the past three decades since 1993, pioneered by Evans, Cohen, Morriss, Searles, Gallavotti, Jarzynski, Crooks, Lebowitz, Spohn, Seifert, et al.

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- The fluctuation theorems provide a generalization of the Second Law of Thermodynamics, in terms of equalities rather than inequalities.
- (Integral) Fluctuation theorems state that for various thermodynamic functionals (denoted as $\mathcal{A}_t(\omega)$) along a stochastic trajectory, such as entropy production, dissipative work, and so on, a generic equality holds :

$$\mathbf{E}e^{-\mathcal{A}_t(\omega)} = 1, \ \forall t,$$

followed by the inequality $\mathbf{E}\mathcal{A}_t(\omega) \geq 0$ (Traditional Second Law).

Fluctuation theorems

- In terms of stochastic process, for each of those $\mathcal{A}_t(\omega)$, there always exists another probability measure $\widehat{P}_{[0,t]}$ for each t which is absolutely continuous with that of the original process $P_{[0,t]}$, such that

$$A_t[\omega] = \ln \frac{dP_{[0,t]}[\omega]}{d\widehat{P}_{[0,t]}[\omega]},$$

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- In all cases, $\widehat{P}_{[0,t]}$ is either (1) the original measure or (2) the time-reversed one of another specific stochastic process.
- Mathematical proof can be found in (Chetrite and Gawedzki, CMP2008). Jarzynski/Hatano-Sasa fluctuation theorems are also corollaries of Feynman-Kac formula (Ge and Jiang, JSP2008).

When Fluctuation Theorems meet multi scales (also possibly with both odd and even variables)

- Langevin-Kramers process

$$\begin{cases} dX_t & = V_t, \\ mdV_t & = -\gamma(X_t)V_t + G(X_t) + \sigma(X_t)dB_t, \end{cases}$$

in which $\sigma^2(x) = 2\gamma(x)k_B T(x)$;

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- In the limit of zero mass, the averaged entropy production rate $e_p(t)$ (Celani, et al. PRL2012 ; Ge, PRE2014)

$$e_p(t) \xrightarrow{m\downarrow 0} e_p^{over}(t) + \frac{5}{6} < \frac{k_B T}{\gamma}(\frac{\nabla T}{T})^2 >,$$

  in which $e_p^{over}(t)$ is averaged entropy production rate of the limiting process, and the additional nonnegative term is called an anomalous contribution.

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  in which $e_p^{over}(t)$ is averaged entropy production rate of the limiting process, and the additional nonnegative term is called an anomalous contribution.
- Convergence of the corresponding functionals along trajectories and the fluctuation theorem of anomalous contribution is also derived (Celani, et al. PRL2012; Bo and Celano, PR2017).

Still lacking...

- General mathematical setup of the problem ;
- What does the anomalous terms generally look like ;
- Rigorous proof using measure theory and multiscale analysis ;
- Sufficient and necessary conditions for the vanishing of anomalous terms.

Multiscale analysis of diffusion processes

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- Here, we consider the infinitesimal generator dependent on parameter ε, including
  (1) the first-order perturbation case, i.e.

$$L^{\varepsilon} = \frac{1}{\varepsilon} L_0 + L_1,$$

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  (2) the second-order perturbation case, i.e.

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- Also called first-order and second-order averaging.

General assumptions

- Assumption 1

  All the drift and diffusion coefficients in the considered SDEs are smooth and satisfy the linear growth condition, and the diffusion coefficients satisfy the uniformly ellipticity condition.

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- Assumption 2

  For each fixed x, t, the null space of L_0 consists only of constants, i.e. $L_0 1(y) = 0$. Here $1(y)$ denotes constants in y. Also there exists a unique solution to the stationary Fokker-Planck equation

  $$L_0^* \rho^\infty(y; x, t) = 0.$$

  L_0^* denotes the adjoint operator of L_0 with respect to Lebesgue measure. $\rho^\infty(y; x, t)$ is the density of an ergodic measure $\mu_{x,t}(dy) = \rho^\infty(y; x, t)dy$.

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  - Sufficient conditions on drift and diffusion coefficients can be found in (Pardoux and Veretennikov, AoP2001,2003,2005).

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- Under Assumptions 1 and 2, for $\varepsilon \to 0$, $X_t^{1,\varepsilon} \Rightarrow X_t^1$, in the sense of weak convergence. X_t^1 satisfies the following stochastic differential equations

$$dX_t^1 = \overline{b}(X_t^1, t)dt + (\overline{D}(X_t^1, t))^{1/2}dB_t,$$

in which

$$\overline{b}(x, t) = \int_Y b(x, y, t)\mu_{x,t}(dy)$$

$$\overline{D}(x, t) = \int_Y \sigma(x, y, t)\sigma(x, y, t)^T \mu_{x,t}(dy).$$

Functional I : the forward case

- Define a first comparable process

$$\begin{cases} d\widehat{X}_t^{1,\varepsilon} & = d(\widehat{X}_t^{1,\varepsilon}, \widehat{Y}_t^{1,\varepsilon}, t)dt + \sigma(\widehat{X}_t^{1,\varepsilon}, \widehat{Y}_t^{1,\varepsilon}, t)d\widehat{B}_t^{(1)}, \\ d\widehat{Y}_t^{1,\varepsilon} & = \varepsilon^{-1}u(\widehat{X}_t^{1,\varepsilon}, \widehat{Y}_t^{1,\varepsilon}, t)dt + \varepsilon^{-1/2}\beta(\widehat{X}_t^{1,\varepsilon}, \widehat{Y}_t^{1,\varepsilon}, t)d\widehat{B}_t^{(2)}, \end{cases}$$

and the probability measure $\widehat{P}_{[0,t]}^{1,\varepsilon}$ can be introduced as the distributions of $\{\widehat{X}_u^{1,\varepsilon}, \widehat{Y}_u^{1,\varepsilon}\}_{0\le u\le t}$ on $(W_1 \times W_2, \mathcal{B}_t)$, in which $W_1 = C([0,\infty), \mathbf{R}^m)$ and $W_2 = C([0,\infty), \mathbf{R}^n)$.

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- The first functional is defined as

$$F_t^{1,\varepsilon}(\omega) = \log \frac{dP_{[0,t]}^{1,\varepsilon}}{d\widehat{P}_{[0,t]}^{1,\varepsilon}}(\omega),$$

  recalling that $P_{[0,t]}^{1,\varepsilon}$ is the distributions of $\{X_u^{1,\varepsilon}, Y_u^{1,\varepsilon}\}_{0\leq u\leq t}$ induced by the original process.

Functional II : the reversal case

- Define a second comparable process

$$\begin{cases} d\widehat{X}_s^{1,\varepsilon,R} & = d(\widehat{X}_s^{1,\varepsilon,R}, \widehat{Y}_s^{1,\varepsilon,R}, t-s)ds + \sigma(\widehat{X}_s^{1,\varepsilon,R}, \widehat{Y}_s^{1,\varepsilon,R}, t-s)d\widehat{B}_s^{(1)}, \\ d\widehat{Y}_s^{1,\varepsilon,R} & = \varepsilon^{-1}u(\widehat{X}_s^{1,\varepsilon,R}, \widehat{Y}_s^{1,\varepsilon,R}, t-s)ds + \varepsilon^{-1/2}\beta(\widehat{X}_s^{1,\varepsilon,R}, \widehat{Y}_s^{1,\varepsilon,R}, t-s)d\widehat{B}_s^{(2)}, \end{cases}$$

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- The second functional for averaging is defined as

$$F_t^{1,\varepsilon,R}(\omega) = \log \frac{dP_{[0,t]}^{1,\varepsilon}}{d\widehat{P}_{[0,t]}^{1,\varepsilon,R}}(\omega).$$

More assumptions

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Assumption 5

$\rho^\infty(y; x, t)$ *satisfies*

$$2G^{-1}(x, y, t)u^{ir}(x, y, t) - G^{-1}(x, y, t)\frac{\partial}{\partial y}G(x, y, t) = \frac{\partial}{\partial y}\log \rho^\infty(y; x, t),$$

in which $u^{ir}(x, y, t) = \frac{1}{2}\left[u_i(x, y, t) + \epsilon_i u_i(\epsilon x, \epsilon y, t)\right]$ *and* $G = \beta\beta^T$.

Anomalous contribution and fluctuation theorems

Theorem 1

Under Assumptions 1-5, the functionals $F_t^{1,\varepsilon}$ and $F_t^{1,\varepsilon,R}$ are both well defined ; the joint process $(X_t^{1,\varepsilon}, F_t^{1,\varepsilon})$ converges to (X_t^1, F_t^1), and $(X_t^{1,\varepsilon}, F_t^{1,\varepsilon,R})$ converges to $(X_t^1, F_t^{1,R})$ weakly in $C([0,\infty), \mathbf{R}^m \times \mathbf{R})$, as $\varepsilon \to 0$. The limiting functionals F_t^1 and $F_t^{1,R}$ can be divided into two terms, i.e.

$$F_t^1 = F_t^{1,a} + F_t^{1,b}, \ F_t^{1,R} = F_t^{1,R,a} + F_t^{1,R,b},$$

where

$$F_t^{1,a}(\omega) = \log \frac{d\overline{P}^1_{[0,t]}(\omega)}{d\widehat{\overline{P}}^1_{[0,t]}(\omega)}, \ F_t^{1,R,a}(\omega) = \log \frac{d\overline{P}^1_{[0,t]}(\omega)}{d\widehat{\overline{P}}^{1,R}_{[0,t]}(\omega)},$$

are the same functionals defined for the averaged process $\{X_t^1\}$. With respect to $(W_1, \mathcal{B}_t^1, \overline{P}^1)$, $e^{-F_t^{1,b}}$ and $e^{-F_t^{1,R,b}}$ are both exponential martingales with mean 1. Necessary and sufficient conditions for the vanishing of $F_t^{1,b}$ and $F_t^{1,R,b}$ are also given.

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$$\begin{cases} dX_t^{2,\varepsilon} & = [\varepsilon^{-1} f(X_t^{2,\varepsilon}, Y_t^{2,\varepsilon}, t) + b(X_t^{2,\varepsilon}, Y_t^{2,\varepsilon}, t)]dt + \sigma(X_t^{2,\varepsilon}, Y_t^{2,\varepsilon}, t) dB_t^{(1)}, \\ dY_t^{2,\varepsilon} & = [\varepsilon^{-1} g(X_t^{2,\varepsilon}, Y_t^{2,\varepsilon}, t) + \varepsilon^{-2} u(X_t^{2,\varepsilon}, Y_t^{2,\varepsilon}, t)]dt + \varepsilon^{-1} \beta(X_t^{2,\varepsilon}, Y_t^{2,\varepsilon}, t) dB_t^{(2)}. \end{cases}$$

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Assumption 6

$f(x, y, t)$ averages to zero under the measure $\mu_{x,t}(dy)$, i.e.

$$\overline{f} = \int f(x, y, t)\mu_{x,t}(dy) = 0,$$

for each x and t.

Then define $\phi(x, y, t)$ as the unique solution of

$$-L_0\phi(x, y, t) = f(x, y, t),$$

with the constrain $\overline{\phi} = 0$.

Second-order perturbation

- Under Assumptions 1,2,6, for $\varepsilon \to 0$, $X_t^{2,\varepsilon} \Rightarrow X_t^2$, in the sense of weak convergence. X_t^2 satisfies the following stochastic differential equations

$$dX_t^2 = F(X_t^2, t)dt + A(X_t^2, t)dB_t,$$

in which

$$F(x,t) = \overline{b(x,y,t) + (f(x,y,t)\cdot\nabla_x + g(x,y,t)\cdot\nabla_y)\phi(x,y,t)},$$

$$A(x,t)A(x,t)^T = A_1(x,t) + \frac{1}{2}(A_0(x,t) + A_0(x,t)^T),$$

$$A_0(x,t) = 2\int_Y f(x,y,t)\otimes\phi(x,y)\rho^\infty(y;x,t)dy,$$
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and the probability measure $\widehat{P}_{[0,t]}^{2,\varepsilon}$ can be introduced as the distributions of $\{\widehat{X}_u^{2,\varepsilon}, \widehat{Y}_u^{2,\varepsilon}\}_{0 \le u \le t}$ on $(W_1 \times W_2, \mathcal{B}_t)$.

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and the probability measure $\widehat{P}_{[0,t]}^{2,\varepsilon,R}$ is introduced as the distributions of $\{\epsilon\widehat{X}_{t-s}^{2,\varepsilon,R}, \epsilon\widehat{Y}_{t-s}^{2,\varepsilon,R}\}_{0\le s\le t}$, where $\epsilon = \{\epsilon_i\}$, $\epsilon_i = \pm 1$ for even and odd variables, respectively.

Functional IV : the reversal case

- Define a second comparable process

$$\begin{cases} d\widehat{X}_s^{2,\varepsilon,R} & = [\varepsilon^{-1} f(\widehat{X}_s^{2,\varepsilon,R}, \widehat{Y}_s^{2,\varepsilon,R}, t-s) + d(\widehat{X}_s^{2,\varepsilon,R}, \widehat{Y}_s^{2,\varepsilon,R}, t-s)]ds \\ & +\sigma(\widehat{X}_s^{2,\varepsilon,R}, \widehat{Y}_s^{2,\varepsilon,R}, t-s) dB_s^{(1)}, \\ d\widehat{Y}_s^{2,\varepsilon,R} & = [\varepsilon^{-1} g(\widehat{X}_s^{2,\varepsilon,R}, \widehat{Y}_s^{2,\varepsilon,R}, t-s) + \varepsilon^{-2} u(\widehat{X}_s^{2,\varepsilon,R}, \widehat{Y}_s^{2,\varepsilon,R}, t-s)]ds \\ & +\varepsilon^{-1}\beta(\widehat{X}_s^{2,\varepsilon,R}, \widehat{Y}_s^{2,\varepsilon,R}, t-s) dB_s^{(2)}. \end{cases}$$

  and the probability measure $\widehat{P}_{[0,t]}^{2,\varepsilon,R}$ is introduced as the distributions of $\{\epsilon\widehat{X}_{t-s}^{2,\varepsilon,R}, \epsilon\widehat{Y}_{t-s}^{2,\varepsilon,R}\}_{0\leq s\leq t}$, where $\epsilon = \{\epsilon_i\}$, $\epsilon_i = \pm 1$ for even and odd variables, respectively.
- The second functional for homogenization is defined as

$$F_t^{2,\varepsilon,R}(\omega) = \log \frac{dP_{[0,t]}^{2,\varepsilon}}{d\widehat{P}_{[0,t]}^{2,\varepsilon,R}}(\omega).$$

More assumptions

Assumption 7

All the ε^{-1} *drift terms are conservative, i.e.*

$$f_i(x, y, t) = -\epsilon_i f_i(\epsilon x, \epsilon y, t), \;\; g_i(x, y, t) = -\epsilon_i g_i(\epsilon x, \epsilon y, t), \;\; \forall i.$$

Assumption 8

For each $i \neq j$,

$$\overline{f_i \phi_j} = 0,$$

recalling that $\phi(x, y, t)$ *is the solution of*

$$-L_0 \phi(x, y, t) = f(x, y, t),$$

with $\overline{\phi} = 0$.

Anomalous contribution and fluctuation theorems

Theorem 2

Under Assumptions 1-8, the functionals $F_t^{2,\varepsilon}$ and $F_t^{2,\varepsilon,R}$ are both well defined; the joint process $(X_t^{2,\varepsilon}, F_t^{2,\varepsilon})$ converges to (X_t^2, F_t^2), and $(X_t^{2,\varepsilon}, F_t^{2,\varepsilon,R})$ converges to $(X_t^2, F_t^{2,R})$ weakly in $C([0,\infty), \mathbf{R}^m \times \mathbf{R})$, as $\varepsilon \to 0$. The limiting functionals F_t^2 and $F_t^{2,R}$ can be divided into two terms, i.e.

$$F_t^2 = F_t^{2,a} + F_t^{2,b},\ F_t^{2,R} = F_t^{2,R,a} + F_t^{2,R,b},$$

where

$$F_t^{2,a}(\omega) = \log \frac{d\overline{P}^2_{[0,t]}(\omega)}{d\widehat{\overline{P}}^2_{[0,t]}(\omega)},\ F_t^{2,R,a}(\omega) = \log \frac{d\overline{P}^2_{[0,t]}(\omega)}{d\widehat{\overline{P}}^{2,R}_{[0,t]}(\omega)},$$

are the same functionals defined for the averaged process $\{X_t^2\}$. With respect to $(W_1, \mathcal{B}_t^1, \overline{P}^2)$, $e^{-F_t^{2,b}}$ and $e^{-F_t^{2,R,b}}$ are both exponential martingales with mean 1. Necessary and sufficient conditions for the vanishing of $F_t^{2,b}$ and $F_t^{2,R,b}$ are also given.

Key points of proof

- Consider the joint process $(X_t^\varepsilon, F_t^\varepsilon)$ as the slow components of $(X_t^\varepsilon, F_t^\varepsilon, Y_t^\varepsilon)$.

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Key points of proof

- Consider the joint process $(X_t^\varepsilon, F_t^\varepsilon)$ as the slow components of $(X_t^\varepsilon, F_t^\varepsilon, Y_t^\varepsilon)$.
- The expression of the functionals in the forward case is directly from the Cameron-Martin-Girsanov formula.
- The expression of the functionals in the reversal case is much more difficult. We combine the Cameron-Martin-Girsanov formula with the standard techniques in measure theory.
  - Find a diffusion process with measure $\widetilde{P}$, with the transition probability density function satisfying

$$\tilde{p}(t, x_2, y_2 | s, x_1, y_1) = \tilde{p}(T - s, \epsilon x_1, \epsilon y_1 | T - t, \epsilon x_2, \epsilon y_2);$$

  - Prove the Radon-Nikodym derivative between $\widetilde{P}$ and $\widetilde{P}^R$;
  - Use the well known Cameron-Martin-Girsanov formula to calculate the Radon-Nikodym derivative between P and $\widetilde{P}$;
  - Use the well known Cameron-Martin-Girsanov formula to calculate the Radon-Nikodym derivative between $\widehat{P}$ and $\widetilde{P}^R$.

Physical Applications

(First-order) Averaging for first-order SDEs

- Consider the first-order stochastic process :

$$\begin{cases} dX_t^\varepsilon & = b(X_t^\varepsilon, Y_t^\varepsilon, t)dt + \sigma(X_t^\varepsilon, Y_t^\varepsilon, t)dB_t^{(1)}, \\ dY_t^\varepsilon & = \varepsilon^{-1} u(X_t^\varepsilon, Y_t^\varepsilon, t)dt + \varepsilon^{-1/2} \beta(X_t^\varepsilon, Y_t^\varepsilon, t)dB_t^{(2)}. \end{cases}$$

Both variables x and y are even variables : the reverse path of $\{X_s, Y_s\}_{0 \le s \le t}$ is $\{X_{t-s}, Y_{t-s}\}_{0 \le s \le t}$.

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  Both variables x and y are even variables : the reverse path of $\{X_s, Y_s\}_{0\leq s\leq t}$ is $\{X_{t-s}, Y_{t-s}\}_{0\leq s\leq t}$.
- Entropy production as a functional in the reversal case

$$S_{tot} = \ln \frac{dP_{[0,T]}(\mathbf{x}, \mathbf{y})}{dP^R_{[0,T]}(\mathbf{x}, \mathbf{y})}.$$

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- Two different origins of entropy production(Ge, PRE2009 ; Ge and Qian, PRE2010).

$$S_{tot} = S_1 + S_2.$$

Different origins of the entropy production

- Adjoint dynamics : another diffusion process with

$$b^{ad} = -b + \nabla_x \sigma\sigma^T + \sigma\sigma^T \nabla_x \log \rho^{ss};$$
$$u^{ad} = -u + \nabla_y \beta\beta^T + \beta\beta^T \nabla_y \log \rho^{ss},$$

in which $\rho^{ss}(x, y, t)$ is the pseudo-invariant distribution at time t.

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- Free energy dissipation

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- Free energy dissipation

$$S_1 = \ln \frac{dP_{[0,T]}(\mathbf{x}, \mathbf{y})}{dP^{ad,R}_{[0,T]}(\mathbf{x}, \mathbf{y})}.$$

- Housekeeping heat

$$S_2 = \ln \frac{dP_{[0,T]}(\mathbf{x})}{dP^{ad}_{[0,T]}(\mathbf{x})}.$$

Anomalous contribution and fluctuation theorem

- Using Theorem 1, for $\varepsilon \to 0$, the joint process

$$(X_t^\varepsilon, S_{tot}^\varepsilon, S_1^\varepsilon, S_2^\varepsilon) \Rightarrow (X_t, S_{tot}, S_1, S_2),$$

in which

$$\begin{aligned} S_{tot} &= \widetilde{S_{tot}} + S_{anom}, \\ S_1 &= \widetilde{S}_1, \\ S_2 &= \widetilde{S}_2 + S_{anom}. \end{aligned}$$

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- The anomalous part S_{anom} is

$$S_{anom} = \int_0^T \frac{1}{2} I dt + \int_0^T I^{1/2} d\xi_t.$$

ξ_t is an independent Wiener process and I is defined by

$$I = \sum_i \left[\overline{D_i^{-1} \left(2b_i - \frac{\partial}{\partial x_i} D_i - D_i \frac{\partial}{\partial x_i} \log \rho^\infty \right)^2} - \overline{D_i}^{-1} \left(2\overline{b_i} - \frac{\partial}{\partial x_i} \overline{D_i} \right)^2 \right].$$

(Second-order) Averaging for second-order SDEs

- Consider the following second-order SDE

$$\begin{cases} dX_t^\varepsilon &= \varepsilon^{-1} V_t^\varepsilon, \\ dV_t^\varepsilon &= [\varepsilon^{-1} g(X_t^\varepsilon, V_t^\varepsilon, t) + \varepsilon^{-2} u(X_t^\varepsilon, V_t^\varepsilon, t)]dt + \varepsilon^{-1}\beta(X_t^\varepsilon, V_t^\varepsilon, t) dB_t. \end{cases}$$

v is an odd variable, hence the reverse path of $\{X_s, V_s\}_{0\le s\le t}$ is $\{X_{t-s}, -V_{t-s}\}_{0\le s\le t}$.

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  v is an odd variable, hence the reverse path of $\{X_s, V_s\}_{0\leq s\leq t}$ is $\{X_{t-s}, -V_{t-s}\}_{0\leq s\leq t}$.
- A remaining part $S_3 = S_{tot} - S_1 - S_2$(Lee, et al. PRL2013)

$$S_3 = \sum_i - \int_0^T \frac{\partial \log \rho^{ss}(X_t^\varepsilon, V_t^\varepsilon; t)}{\partial v_i} \circ dV_t^{\varepsilon,i} - \int_0^T \frac{\partial \log \rho^{ss}(X_t^\varepsilon, V_t^\varepsilon; t)}{\partial x_i} dX_t^{\varepsilon,i} + \int_0^T \frac{\partial \log \rho^{ss}(X_t^\varepsilon, -V_t^\varepsilon; t)}{\partial v_i} \circ dV_t^{\varepsilon,i} + \int_0^T \frac{\partial \log \rho^{ss}(X_t^\varepsilon, -V_t^\varepsilon; t)}{\partial x_i} dX_t^{\varepsilon,i},$$

  arising from the asymmetry of steady state distribution with respect to the odd variable v.

Anomalous contribution and fluctuation theorem

- Using Theorem 2, for $\varepsilon \to 0$, the joint process

$$(X_t^\varepsilon, S_{tot}^\varepsilon, S_1^\varepsilon, S_2^\varepsilon, S_3^\varepsilon) \Rightarrow (X_t, S_{tot}, S_1, S_2, S_3),$$

in which

$$\begin{aligned} S_{tot} &= \widetilde{S_{tot}} + S_{anom}, \\ S_1 &= \widetilde{S}_1, \\ S_2 &= \widetilde{S}_2 + S_{anom}, \\ S_3 &= 0. \end{aligned}$$

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$$S_{anom} = \int_0^T \frac{1}{2} l dt + \int_0^T l^{1/2} d\xi_t.$$

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- The anomalous part S_{anom} is

$$S_{anom} = \int_0^T \frac{1}{2} I dt + \int_0^T I^{1/2} d\xi_t.$$

- In the case $u = -\gamma V_t$ and $\beta = \sqrt{2T(X_t)\gamma}$, our results reduce to the anomalous contribution in (Celani, et al. PRL2012 ; Ge, PRE2014 ; Bo and Celani, PR2017).

$$I = \sum_i \frac{n+2}{3\gamma T(x)} \left(\frac{\partial T(x)}{\partial x_i} \right)^2.$$

Summary

Original process

$$\{(X_t^\varepsilon, Y_t^\varepsilon)\}$$

$$
\begin{array}{lccccc}
 & \{X_t^\varepsilon\} & \longrightarrow & \{X_t\} & & \\
 & \downarrow & & \downarrow & & \\
\text{Functionals} & F_t^\varepsilon & \stackrel{\varepsilon\to 0}{\longrightarrow} & F_t^1 & + & F_t^2 \\
 & \uparrow & & \uparrow & & \uparrow \\
 & \{\widehat{X}_t^\varepsilon\} & \longrightarrow & \{\widehat{X}_t\} & & \text{Anomalous contribution}
\end{array}
$$

Comparable process

$$\{(\widehat{X}_t^\varepsilon, \widehat{Y}_t^\varepsilon)\}$$

$$\mathbf{E}e^{-F_t^\varepsilon} = \mathbf{E}e^{-F_t^1} = \mathbf{E}e^{-F_t^2} \equiv 1, \ \forall t \geq 0.$$

Ongoing work

- $\mathbf{E}e^{-F_t} = 1$, in which $F_t = F_t^1 + F_t^2$?
- Dissipative work of Jarzynski equality is included in our frameworks ?
- Structure of the decomposition $F_t = F_t^1 + F_t^2$, orthogonal ? in which sense ?
- Fluctuation theorems under homogenization (spatial averaging).

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Thank you for your attention !